FOREWORD

BY BARRY MAZUR

It is exhilarating to think about the very, very large.

One of the pleasures of mathematics is that it allows us to go even further, to *go all the way,* in fact, and imagine the *infinite,* a concept that we might naively have felt was beyond imagination. *Infinity: Beyond the Beyond the Beyond,* written by Lillian Lieber with drawings by Hugh Gray Lieber, is an exuberant celebration of this pleasure.

The *Beyond*

A friend of mine opened a session of an after-school math class for very young children with the question "What is the largest number?" One of the children, without a moment's hesitation, offered "Nine thousand nine hundred." A savvier child piped up with the challenge, "Then what about nine thousand nine hundred and one?" "OK . . . I guess I was wrong," conceded the first child. "But I was close!"

The same friend, in teaching geometry to such a class, once began his discussion with the invitation, "Think of a straight line," and was stopped short, halfway through his next sentence, by one of his students. "Wait!" the child said. "I'm still thinking of it."

The Liebers would have instantly fallen in love with those youngsters grappling with the inexhaustibility of the math. For our authors are veritable wrestlers with infinity: they adore the sport. They try to surprise infinity, to sneak up to it, and then somehow exult in their failure to capture it. They impress themselves, and us, with some large number (molecules of a gas in a given volume, grains of sand on a beach, electrons in the universe) and then dismiss the humongous number with the swipe, "but it is NOT INFINITY by any means."

At present writing, the largest explicitly known prime number is

$$2^{25964951} - 1,$$

a hefty thing weighing in with 7,816,230 decimal digits. Nevertheless I'm sure that the Liebers—if confronted with it—would straightaway compare this behemoth to their sublime INFINITY and find it sorely wanting.

Our gallant authors are not defeated though, in their quest for the infinite, despite the paltry finiteness of any number that can be digitalized and exhibited. They pass to geometry where, for example, by slicing a conic section in various ways, they offer us a moving picture of taffy-long, and even longer, ellipses that finally burst into a parabola as one of their points scoots out to—yes—"infinity."

Beyond *The Beyond*

Soon, though, the Liebers get to the infinite *thing-in-itself:* namely, Cantor's arithmetic of infinities, his theory of infinite cardinals and ordinals.*

*The Church of England has its ordinals—the Roman church, its Cardinals—and the Hebrew bible its alephs. This ecumenical brew all comes together as nomenclature in Cantor's theory.

Cantor had the genius to perform the seemingly innocuous sleight-of-mind of replacing the act of *counting* by, in effect, what is its near-equivalent: the act of *putting two sets in one-one correspondence.** With that, he became the voyager to discover a new world of ideas. For he found that there are infinite sets *A, B* of different *sizes,* in the sense that the members of *A* cannot be put in one-one correspondence with the members of *B.* There are, then, *larger* infinities, and *smaller* ones. And, moreover, these distinct "infinities" behave according to the laws of a stunning arithmetic.

The vast difference between *counting* (i.e., one, two, three, . . .) and *ordering* (i.e., first, second, third, . . .) emerges in this infinite arithmetic; the first activity yields *cardinal numbers* and the second *ordinal numbers.* In the finite realm these two concepts are distinguished only by intent, perhaps—but not by extent. However, they show themselves to be quite, quite different in the infinite world.

Cantor's theory of sets—or at least the general formulation of "set theory"—is currently the lingua franca of modern mathematics, for it is the rare new concept that is introduced onto the stage of mathematics nowadays without some mention of the vocabulary related to sets. This is quite a triumph for *the theory of sets,* and especially so in view of the continuing perplexity of what it *means* to

* E.g., as in removing the word "number" from the sentence:

> There are the same number of seats in this auditorium as people in it.

and rephrasing it as:

> Every seat in this auditorium is occupied and no one is standing.

be a set. The Liebers—their enthusiasm (for the infinite) unbounded—edge into the famous controversies that then surrounded Cantor's theory. They cite another enthusiast, the great mathematician David Hilbert, who referred to Cantor's work on transfinite numbers as "the most wonderful flowering of the spirit of mathematics, and indeed one of the greatest achievements of human reason."

Beyond *The Beyond The Beyond*

The Liebers mention the *continuum question,* one of the extraordinary legacies of Cantor: *Is there a cardinal larger than the cardinal of the set of counting numbers* $\{1, 2, 3, \ldots\}$ *and smaller than the cardinal of the set of all real numbers?* which was an open problem when they were writing their book. It would tell us much about our authors if we knew what their reaction was to the ultimate astounding answer to this continuum question, obtained by Paul Cohen after their book had been written.

The gist of Cohen's discovery is that *it depends on what you mean by "set."* Specifically, there are models of *set theory* (i.e., models fulfilling *the* standard collection of axioms for set theory) that go either way: There is a model in which cardinals exist that are strictly between the cardinal of the set of counting numbers $\{1, 2, 3, \ldots\}$ and that of the set of all real numbers, i.e., that are larger than the first cardinal and smaller than the second; and there is another model in which such in-between cardinals do not exist. Kurt Gödel, I imagine, would simply interpret this development as offering us all the more reason to formulate our systems of axioms for set theory "better," so that only models of the latter type are allowed.

Infinity Abridged

All times have their troubles, and the epoch during which this little book was written—just after the Second World War—certainly came with its share of trepidations, some of which (the efficacy of the fledgling United Nations, the future uses of atomic energy) somehow made their appearance in this book.

The *joy of thinking* that the Liebers radiate is timeless, but some of their more worldly concerns and some of their choices of mathematical topics speak less to a modern audience than they did to their readers in 1953. So I have made some judicious abridgments, in order to guide you to the best of what the Liebers have to offer—the mathematics of transfinites. To that end I deleted the preface and chapter 1 of their original version, in which the Lieber's spirit-guide SAM is introduced. (I will describe him, briefly, below.) I also deleted their chapters 18 through 24 and part of chapter 17, which comprised a treatment of the calculus of Newton and Leibniz and modern extensions thereof. I felt that this part of their work was less focused on the "beyond the beyond the beyond," the grand mission of the earlier chapters, and appeared as a lengthy, separable, coda. Since there are many treatments of this material more approachable by a modern audience, its deletion intensifies the focus of Lillian Lieber's text.*

Readers of the Liebers' earlier classic work *The Education of T. C. MITS* will be acquainted with the eponymous Mr. MITS (*The Celebrated Man In The Street*). But they will

* I have also removed footnotes when they cited texts that are no longer relevant or up to date.

not yet have met the dedicatee of this book, whom Lillian Lieber refers to simply as SAM and who is introduced in parts of the text that are omitted in this edition. Here, then, is a brief introduction to SAM.

SAM is more a *spirit* that can inhabit a being, rather than a *being* proper, and is the conjunction of the munificence of **S**cience, **A**rt, and **M**athematics, pitted against whatever darker elements of history and exigency lurk. SAM engages the better parts of our inquisitiveness, our sensibility, and our intellect, and recognizes the pitfalls of humanity. But SAM shines through it all, offering us his bounty of ideas and imagination. Cheerful spirit-guide that he is, SAM knows that he will prevail, and . . . knows that he will keep his sense of humor, as well.

Infinity Enjoyed

We readers are exhorted by the Liebers to judiciously test any idea before adopting it, but then we are reminded—once we've accepted the idea—to rejoice in it as well. The Liebers sometimes appear to me to be discerning shoppers in the platonic fruit-stall of mathematics. Never content just to *think* a concept, they also have to test it, squeeze it, pinch it, sniff it, take a few bites, and muse about their own reactions to it before buying it. And *then,* to thoroughly enjoy it.